Discrete Mathematics: Combinatorics and Graph Theory

Homework 1 Solution

Instructions. Solve any 10 questions. Typeset or write neatly and show your work to receive full credit.

1. Tell us a little bit about yourself. Why are you majoring in computer science (or something else)? What is the relationship between mathematics and computer science? Why are you required to take *discrete mathematics*? Why is there an emphasis on problem solving and proofs rather than on memorizing how to perform computations? What do you hope to get out of the course?

2. Prove that for every integer x and for every integer y, if x is odd and y is odd then xy is odd. Translate into symbols using quantifiers.

   If x is odd, then by definition $\exists\ a \in \mathbb{Z}$ such that $x = 2a + 1$. Similarly y can be written as $2b + 1$. Taking their product: $xy = (2a + 1)(2b + 1) = (4ab + 2a + 2b + 1) = 2(ab + a + b) + 1$. By definition xy is odd since xy can be written as $2c + 1$ with $c = (ab + a + b) \in \mathbb{Z}$.

3. Given a real number x, let A be the statement $\frac{1}{2} < x < \frac{5}{2}$, let B be the statement $x \in \mathbb{Z}$, let C be the statement $x^2 = 1$, and let D be the statement $x = 2$. Which statements below are true for all $x \in \mathbb{R}$?

   (a) $A \to C$
   False. Every number in (1/2,5/2) other than 1 is a counterexample.

   (b) $B \to C$
   False. Every integer not in $\{1, -1\}$ is a counterexample.

   (c) $(A \wedge B) \to C$
   False. The hypothesis is satisfied by 1 and 2, but the conclusion is not satisfied by 2.

   (d) $(A \wedge B) \to (C \vee D)$
   True. The hypothesis is satisfied by 1 and by 2. Since 1 satisfies C and 2 satisfies D, each satisfies the conclusion.

   (e) $C \to (A \wedge B)$
   False. The set of numbers satisfying the hypothesis is $\{1, -1\}$. Among these, both satisfy B, but -1 does not satisfy A. Thus -1 is a counterexample

   (f) $D \to (A \wedge B \wedge \neg C)$
   True. The hypothesis is satisfied only by the number 2. This is in the set $(1/2, 5/2)$, is an integer, and does not yield 1 when squared, so it satisfies the conclusion making the conditional true.

   (g) $(A \vee C) \to B$
   False. The hypothesis is satisfied by -1 and by all numbers in the interval $(1/2, 5/2)$. Of these, only -1,1, and 2 are integers; all other numbers in the interval are counterexamples.

4. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8". Determine whether the following statements are true:

   (a) $(\forall x \in \mathbb{Z})[P(x) \to Q(x)]$
   True. Consider an integer x, under the hypothesis (x is odd), we have $x = 2k + 1$ for some $k \in \mathbb{Z}$. Hence $x^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1)$. If $k \in \mathbb{Z}$, one of k and $k + 1$ is even, hence the product $k(k + 1)$ is even and $4k(k + 1)$ can be written as $4 \times 2m = 8m$ is divisible by 8.

   (b) $(\forall x \in \mathbb{Z})[Q(x) \to P(x)]$
   True. Consider the contrapositive $\neg P(x) \to \neg Q(x)$. If x is not odd, then x is even, so x^2 is even, and hence $x^2 - 1$ is odd. Therefore $x^2 - 1$ and not divisible by 8.

Let P(x) be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer". Determine whether the following statements are true:

(a) $(\forall x \in \mathbb{Z})[P(x) \to Q(x)]$
False. Any odd integer is a counterexample since the hypothesis is true and the conclusion is false.

(b) $(\forall x \in \mathbb{Z})(P(x)) \to (\forall x \in \mathbb{Z})(Q(x))$
True. The hypothesis (all integers are odd) is false, as is the conclusion (all integers are even). Since the hypothesis is false, the conditional is true regardless of whether the conclusion is true.

5. Prove the following by induction. State base case, inductive hypothesis and inductive step to receive full credit.

(a) The sum of the first n odd natural numbers equals n^2.
Base case: $1 = 1^2$. *Inductive hypothesis*: Assume $1 + 3 + \cdots + (2k-1) = k^2$. *Inductive step*: $1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2$. $\square$

(b) $\forall n \in \mathbb{N}$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Base case: $1^2 = \frac{1(2)(3)}{6}$. *Inductive hypothesis*: Assume $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. *Inductive step*: $1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)[n(2n+1)+6(n+1)]}{6} = \frac{(n+1)[2n^2+7n+6]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$. $\square$

(c) $\forall n \in \mathbb{N}$, $n^2 - n$ is even.
Base case: (2^2-2) is even. *Inductive hypothesis*: (k^2-k) is even. *Inductive step*: $(k+1)^2-(k+1) = k^2 + k = k(k+1)$. By definition, either k or $k+1$ is even, therefore $k(k+1)$ is even. $\square$

(d) $\forall n \in \mathbb{N}$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.
Base case: $1^3 = \frac{1^2(1+1)^2}{4} = 1$. *Inductive hypothesis*: $1^3 + 2^3 + \cdots k^3 = \frac{k^2(k+1)^2}{4}$ for some k. *Inductive step*: $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2+4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k+2)^2}{4}$ $\square$

(e) $\forall n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$.
Base case: $1 \cdot 2 = \frac{1(2)(3)}{3} = 2$. *Inductive hypothesis*: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$. *Inductive step*: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2) = \frac{k(k+1)(k+2)}{3} + (k+1) \cdot (k+2) = \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$. $\square$

6. Injections, surjections and bijections.

(a) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ where (i) f is one-to-one but not onto; and (ii) f is onto but not one-to-one.
$f(x) = 2x$, $f(x) = \lfloor x/2 \rfloor$

(b) For $A = \{1, 2, 3, 4, 5, 6, 7\}$, how many bijective functions $f : A \to A$ satisfy $f(1) \neq 1$? What if $A = \{x | x \in \mathbb{Z}^+, 1 \leq x \leq n\}$ for some fixed $n \in \mathbb{Z}^+$?
There are 7! bijections from $A \to A$ and 6! satisfy $f(1) = 1$, therefore there are $7! - 6! = 6 \cdot 6!$ functions where $f(1) \neq 1$. In general there are $n! - (n-1)! = (n-1)(n-1)!$ for some fixed n.

(c) For $A = (-2, 7] \subseteq \mathbb{R}$ define the functions $f, g : A \to \mathbb{R}$ by $f(x) = 2x - 4$ and $g(x) = \frac{2x^2-8}{x+2}$. Verify that $f = g$. Is the result affected if we change A to $[-7, 2)$?
f, g share the same domain and codomain, and $\forall x \in A$:

$$g(x) = \frac{2x^2 - 8}{(x+2)} = \frac{2(x^2-4)}{(x+2)} = \frac{2(x+2)(x-2)}{(x+2)} = 2(x-2) = 2x - 4 = f(x)\,.$$

When A is changed to to $[-7, 2)$ there is a problem and $f \neq g$. For any nonempty subset $A \subseteq \mathbb{R}$ when $-2 \in A$ then g is not defined for A since $g(-2) = 0/0$.

7. Define a function $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ by $f(p, q) = \frac{p}{q}$. Is f (i) injective, (ii) surjective, and (iii) bijective? Prove or disprove.

   (i) $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ is not injective since $f(1, 1) = f(2, 2) = 1$.

   (ii) f is surjective since every rational number by definition is a quotient of integers. We can always ensure the denominator is positive.

   (iii) f is not a bijection since f is not injective.

8. Verify whether the function $g : \mathbb{R} \to (0, 1)$ where $g(x) = \frac{1}{1+e^{-x}}$ defines a bijection. Verify whether the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ with $f(x) = tan(x)$ defines a bijection.

   (a) Suppose $x_1, x_2 \in \mathbb{R}$ and $g(x_1) = g(x_2)$. Then $g(x_1) = g(x_2) \Rightarrow (1 + e^{-x_1})^{-1} = (1 + e^{-x_2})^{-1} \Rightarrow (1 + e^{-x_1}) = (1 + e^{-x_2}) \Rightarrow e^{-x_1} = e^{-x_2} \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$. Thus $g(x)$ is one-to-one.

   (b) $y = \frac{1}{1+e^{-x}} \Rightarrow (1 + e^{-x}) = \frac{1}{y} \Rightarrow e^{-x} = \frac{1-y}{y} \Rightarrow x = -ln(\frac{1-y}{y}) = ln(\frac{y}{1-y})$. Check $g(x) = (1 + e^{-x})^{-1} = (1 + e^{ln(\frac{1-y}{y})})^{-1} = (1 + \frac{1-y}{y})^{-1} = \frac{y+(1-y)}{y})^{-1} = (1/y)^{-1} = y$. Thus $g(x)$ is onto.

   (c) Suppose $x_1, x_2 \in (-\pi/2, \pi/2)$ and $f(x_1) = f(x_2)$. Then $f(x_1) = f(x_2) \Rightarrow \frac{sin(x_1)}{cos(x_1)} = \frac{sin(x_2)}{cos(x_2)} \Rightarrow sin(x_1)cos(x_2) = sin(x_2)cos(x_1) \Rightarrow sin(x_1)cos(x_2) - sin(x_2)cos(x_1) = 0 = sin(x_1 - x_2)$. Given that $x_1, x_2 \in (-\pi/2, /pi/2) \Rightarrow x - y \in (-\pi, \pi)$. Since $sin(\theta) = 0$ only happens when $\theta = 2\pi n \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. Thus $f(x)$ is one-to-one.

   (d) To show that $tan(x)$ is surjective we need a bit of Calculus. Given that $cos(\pi/2) = 0$ and $sin(\pi/2) = 1$, the limit as x approaches $\pi/2^-$ of $tan(x) = \infty$ and the limit as x approaches $\pi/2^+$ of $tan(x) = -\infty$. The Intermediate Value Theorem implies that the image of tan must be $(-\infty, \infty)$.

9. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined below. Prove that f is one-to-one and onto, and determine f^{-1}.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}$$

   (a) Suppose $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$. Then either $f(x_1), f(x_2)$ are both even or they are both odd. If they are both even, then $f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$. Otherwise, they are both odd and $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Thus f is one-to-one.

   (b) In order to prove that f is onto, let $n \in \mathbb{N}$. If n is even, then $(-n/2) \in \mathbb{Z}$ and $(-n/2) < 0$, and $f(-n/2) = -2(-n/2) = n$. For the case where n is odd we find that $(n + 1)/2 \in \mathbb{Z}$ and $(n + 1)/2 > 0$, and $f((n + 1)/2) = 2((n + 1)/2)) - 1 = (n + 1) - 1 = n$. Hence f is onto.

   (c) $f^{-1} : \mathbb{N} \to Z$ is defined:

$$f^{-1}(x) = \begin{cases} \frac{(x+1)}{2} & \text{if } x = 1, 3, 5, \cdots \\ -x/2 & \text{if } x = 0, 2, 4, \cdots \end{cases}$$

10. Construct an explicit bijection from the open interval (0, 1) to the closed interval [0, 1].
We need to handle the points 0 and 1. Let $f : (0, 1) \to [0, 1]$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ \frac{1}{n-2} & \text{if } x = 1/n\,,\, n > 2 \\ x & \text{if } x \in (0, 1) - \{\frac{1}{n} | n \geq 2\} \end{cases}$$

To show that $f(x)$ is injective:

   (a) $n_1, n_2 > 2$ so that $x_1 = \frac{1}{n_1}, x_2 = \frac{1}{n_2}$. $f(x_1) = f(x_2) \Rightarrow \frac{1}{n_1-2} = \frac{1}{n_2-2} \Rightarrow n_1 - 2 = n_2 - 2 \Rightarrow n_1 = n_2 \Rightarrow \frac{1}{n_1} = \frac{1}{n_2} \Rightarrow x_1 = x_2$.

   (b) If $x \in (0, 1) - \{\frac{1}{n} | n \geq 2\}$, then $f(x_1) = x_1$ and $f(x_2) = x_2$. When $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Both cases produce different images for different elements, hence $f(x)$ is one-to-one.

The point $0 \in [0,1]$ is mapped onto since by construction $f(1/2) = 0$. The point $1 \in [0,1]$ is mapped onto since by construction $1/3 \in (0,1)$ and we find $f(1/3) = 1$. For any $x \in (0,1)$, we can find a pre-image y in $(0,1)$ such that $f(y) = x$. Therefore $f(x)$ is onto.

11. Prove the following proposition (hint: use the contrapositive). Suppose a is an integer. If a is odd, then $x^2 + x - a^2 = 0$ has no integer solution.
If $x^2 + x - a$ has an integer solution, then it can be factorized as $x^2 + x - a = (x - n)(x + m)$ where $nm = a^2$ and $m - n = 1$. We can rewrite the second point as $m = n + 1$ and use this to deduce that $a^2 = n(n+1)$. However, we know that $n(n+1)$ is even by definition and therefore cannot be equal to a^2 which is odd. $\square$

12. Prove the following proposition (hint: use proof by contradiction). There does not exist a smallest positive rational number.
Assume there is a smallest positive rational number x. Then $\exists\ a, b \in \mathbb{Z}^+$ such that $x = \frac{a}{b}$ and $b \neq 0$. Define $y = \frac{1}{2} \times \frac{a}{b} = \frac{a}{2b} < \frac{a}{b} = x$ for a contradiction. $\square$

13. Prove the following:
    (a) $\sqrt{5}$ is irrational.
    See lecture notes for a proof that $\sqrt{p}$ is irrational for any prime.
    (b) $\sqrt{20}$ is irrational.
    $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$ which is irrational by (a).
    (c) $\sqrt{2} + \sqrt{5}$ is irrational.
    Assume that $x = \sqrt{2} + \sqrt{5}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that $\sqrt{2} + \sqrt{5} = \frac{a}{b} \Rightarrow \sqrt{5} = \frac{a}{b} - \sqrt{2} \Rightarrow 5 = \frac{a^2}{b^2} - 2\sqrt{2}\frac{a}{b} + 2 \Rightarrow \sqrt{2} = -\frac{3b}{2a} + \frac{a}{2b}$. However this implies that $\sqrt{2}$ is rational which is a contradiction.
    (d) There are infinitely many composite numbers.
    Take $S = \{2n | n \in \mathbb{N}\}$. Any integer (prime or composite) multiplied by 2 has at least two distinct factors. It follows that S is an infinite set of composite numbers.

14. Describe a correspondence between binary real numbers $x = (.b_1 b_2 b_3 \cdots)_2$ in $[0, 1)$ and Stern-Brocot real numbers $\alpha = B_1 B_2 B_3 \cdots$ in $[0, \cdots, \infty)$. If x corresponds to α and $x \neq 0$, what number corresponds to $1 - x$? Give a simple rule for comparing rational numbers based on their representations as L's and R's in the Stern-Brocot number system.
$1/\alpha$. To get $1-x$ from x in binary, we interchange 0 and 1; to get $1/\alpha$ from α in Stern-Brocot notation, we interchange L and R.

15. The Stern-Brocot representation of π is $\pi = R^3 L^7 R^{15} L R^{292} L R L R^2 L R^3 L R^{14} L^2 R \cdots$. Use it to find the simplest rational approximations to π whose denominators are less than 50. Is $\frac{22}{7}$ one of them?
We need to only use the first part of the representation:
$$R\ R\ R\ L\ L\ L\ L\ L\ L\ L\ R\ R\ R\ R\ R\ R$$
$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}, \frac{25}{8}, \frac{47}{15}, \frac{69}{22}, \frac{91}{29}, \frac{113}{36}, \frac{135}{43}$$
The fraction 4/1 appears because it is a better upper bound than 1/0, not because it is closer than 3/1. Similarly, 25/8 is a better lower bound than 3/1. The simplest upper bounds and the simplest lower bounds all appear, but the next really good approcimation doesn't occur until just before the string of R's switches back to L.

16. Show that the union of a countable number of countable sets is countable.
Let $\mathcal{A}$ be a countable set of sets, each of which is countable. We need to show the following is countable:
$$B = \bigcup_{A \in \mathcal{A}} A$$

Since $\mathcal{A}$ is countable, there is a bijection $f : \mathbb{N} \to \mathcal{A}$. Let $A_i = f(i)$. Observe that since each A_i is also countable, for each i there is a surjection $g(i) : \mathbb{N} \to A_i$. We will define a function h that takes the pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ to the j^{th} element of the i^{th} set. Let $h : \mathbb{N} \times \mathbb{N} \to B$ be the function $h(i, j) = g_i(j)$.

Suppose $x \in B$. By the definition of B as a union of sets, $\exists A \in \mathcal{A}$ such that $x \in A$. Since f is a bijection, f is onto and there exists $i \in \mathbb{N}$ such that $g_i(j) = x$. By definition $h(i, j) = g_i(j) = x$ therefore h is onto.

We can now construct a bijection $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ to compose $h \circ \phi : \mathbb{N} \to B$ using the result proved in lecture that the composition of bijections is a bijection. Let us encode a natural number into two natural numbers as follows. Define $n = 2^{r-1}(2s - 1)$ where $r, s \in \mathbb{N}$. By the Fundamental Theorem of Arithmetic there, exists a unique $(r_n, s_n) \in \mathbb{N} \times \mathbb{N}$ such that $n = 2^{r_n-1}(2s_n - 1)$. Letting $phi(n) = (r_n, s_n)$ yields the desired result.

17. Use Cantor's diagonalization argument to show that the Cantor set is uncountable.

    We will use the result from lecture that we can write each $x_i \in \mathcal{C}$ as a ternary expansion $x_i = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$ where $\forall a_i \,, a_i \in \{0, 2\}$.

    By way of contradiction, assume $\mathcal{C}$ is countable with points that can be enumerated $\mathcal{C} = \{x_1, x_2, x_3, \cdots\}$. Consider the digits of each enumerated point $x_i = 0.a_{i1}a_{i2}a_{i3}\cdots$. The points in $\mathcal{C}$ can be listed as follows:

$$
\begin{aligned}
x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\
x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\
x_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\
x_4 &= 0.a_{41}a_{42}a_{43}a_{44}\cdots \\
&\vdots
\end{aligned}
$$

    We will now construct a new point that should be valid but does not appear on the list. Let $d = (d_1, d_2, d_3, \cdots)$ be defined as $d_j = 0$ if $a_{ii} = 2$ and $d_j = 2$ if $a_{ii} = 0$. By construction $d_j \neq a_{ii}$, however for $\mathcal{C}$ to be countable, d should appear in the enumerated list. Hence we have a contradiction and $\mathcal{C}$ is uncountable.

18. Is 1/108 in the Cantor set? What about $\pi/12$? Prove or disprove.

    Yes. Use the result from lecture that $x \in \mathcal{C}$ iff $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$ and $\forall a_i \,, a_i \in \{0, 2\}$. Then $1/108 = (0.009\overline{259})_{10} = (0.000\overline{20})_3$. No. $\pi/12 \in (7/27, 8/27)$ which is removed in $\mathcal{C}_3$.

19. Show that if S is a set, there does not exist an onto function f from S to $\mathcal{P}(S)$ (the power set of S). Conclude that $|S| < |\mathcal{P}(S)|$. Suppose such a function f existed. Let $T = \{s | s \in S, s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$. This is known as Cantor's Theorem.

    We need to show that there exists a one-to-one correspondence between S and a subset of $\mathcal{P}(S)$, but that there is no one-to-one correspondence between $\mathcal{P}(S)$ and S. Let us define $f : S \to \mathcal{P}(S)$ as $f(s) = \{s\}$. Then clearly is one-to-one.

    We need to show that $|S| < |\mathcal{P}(S)|$. By way of contradiction, assume there is a bijection $g : S \to \mathcal{P}(S)$. There exist elements of S that map to subsets of S of which they are a member, and elements of S that mp into a subset of S of which they are not a member. Define N as the set of elements of S that do not map into any subset that they belong to, meaning $N = \{x \in S \cap x \notin g(x)\}$. Given that N is a subset of S and g is one-to-one, there must be an element k such that $k = g(k)$. However, this produces a contradiction. If k belongs to N it cannot belong to N. Similarly, if k does not belong to N then it must belong to N. Therefore the existence of a bijection $g : S \to \mathcal{P}(S)$ produces a contradiction and must be false. We then have that $|S| < |\mathcal{P}(S)|$. Surprisingly for any cardinality, there is always a cardinality of higher order.

20. We defined Stern's diatomic series as $a_0 = 0, a_1 = 1, a_{2n} = n, a_{2n+1} = a_n + a_{n+1}$. Show that the function $f(n) = \frac{a_n}{a_{n+1}}$ defines a bijection from $\mathbb{Z}^+$ to $\mathbb{Q}^+$.

    The following proof from an article by Sam Northshield in the American Mathematical Monthly is easy to digest and uses a modification of the Euclidean Algorithm. We will demonstrate that every relatively prime pair [a,b] appears exactly once in the list

    $$\mathcal{L} = [1,1], [1,2], [2,1], [1,3], [3,2], [2,3], [3,1], \cdots, [a_n, a_{n+1}], \cdots$$

    Consider a *slow* modification to the Euclidean Algorithm (the SEA) on pairs of positive integers. For a given [a,b] pair, subtract the smaller number from the larger number repeatedly until equal. For example,

    $$[4,7] \to [4,3] \to [1,3] \to [1,2] \to [1,1]$$

    The algorithm preserves the gcd(a,b) at each step and always terminates with $[g, g]$ where $g = gcd(a, b)$. Let $\mathcal{L}_n = [a_n, a_{n+1}]$. For any $n > 1$, we then have that $[\mathcal{L}_{2n}, \mathcal{L}_{2n+1}] \to \mathcal{L}_n$. Note that if the SEA: $[a, b] \to \mathcal{L}_n$ then either $[a, b] = \mathcal{L}_{2n}$ or $[a, b] = \mathcal{L}_{2n+1}$. Also, since $\mathcal{L}_1 = [1, 1]$, every $\mathcal{L}_n$ is a relatively prime pair.

    By way of contradiction, suppose there exists a relatively prime pair $[a, b]$ not in $\mathcal{L}$. Then all of its successors under the SEA algorithm including [1,1] are not in $\mathcal{L}$, however this is a contradiction. Therefore, every relatively prime pair appears in $\mathcal{L}$ and $f(n)$ is onto.

    By way of contradiction, suppose a prime pair appears more than once in $\mathcal{L}$. Then there must exist a smallest $n > 1$ such that $\mathcal{L}_n = \mathcal{L}_m$ for some $m > n$. Appling one step of the SEA to both $\mathcal{L}_n$ and $\mathcal{L}_m$ forces $\lfloor n/2 \rfloor = \lfloor m/2 \rfloor \Rightarrow m = n + 1$. Hence $a_n = a_{n+1} = a_{n+2}$ which is a contradiction. Therefore, $f(n)$ is one-to-one and onto.